Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year		Final Term Exam Date: May 30, 2015 Course: Mathematics 1 – B Duration: **3** hours

- Answer **All** Questions The Exam consists of one page
- No. of questions: 4 Total Mark: 100

Question 1

(a)Complete: (i)If $\lambda_1, \lambda_2, \lambda_3$ are all eigenvalues of a matrix A. Then |A| is…. — 2

(ii)A linear system AX = B is called consistent if….

(b)If $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 3 & -1 \end{bmatrix}$. Find, if possible, A + B, A.A\`, B.A, A + 2I, |A| — 8

(c) If $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$. Find (i)The eigenvalues and eigenvectors of A — 8

(ii)The eigenvalues of $f(A) = \ln(A^2 + A)$ (iii) A^n — 2+5

Question 2

(a)Solve the L.S : 2x – y + z = 1, 4x + 2y + 2z = 3, – 2x + 3y – z = 0, x + z = 2 — 4

(b)Find S , S_{10} from: (i) $\sum_{r=1}^{n}(r+1)(2r+1)$ (ii) $\sum_{r=1}^{n} \frac{2}{r^2+4r+3}$ — 6

(c)Using the mathematical induction, prove that:

(i) $1.1! + 2.2! + 3.3! + \cdots + n.n! = (n+1)! - 1$ (ii) $y^{(n)} = \sin(x + \frac{n\pi}{2})$ where $y = \sin x$ — 8

(d)If $z_1 = 1 + i,\ z_2 = -2 + 2i$. Find $z_1 + z_2$, $z_1.z_2$, $\sqrt[2]{z_1}$, $(z_2)^7$ — 5

(e)Using the binomial theorem, expand $\frac{1}{1-2x+x^2}$. — 2

Question 3

(a)Determine the locus of the middle points of chords of the parabola $y^2 = 4ax$ which passes through vertex (0, 0). — 6

(b)Prove that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines if $h^2 = ab$ and $af^2 = bg^2$. — 6

(c)Find the equation of the bisectors for: $2x^2 + xy - 3y^2 - 6x - 14y - 8 = 0$ — 6

(d)Show that the equation $x^2 + 4y^2 + 4x + 8y - 8 = 0$ represents an ellipse. — 7

Determine its major, minor axis, foci and its vertices.

Question 4

(a)Derive the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which bisected at the point (5, 2). — 7

(b)Find the equation of the circle which passes through the points (8, 9), (1, 2) and cut the circle $x^2 + y^2 = 25$ at a right angle. — 6

(c)Trace the curve : $y^2 + 4x + 2y - 11 = 0$. — 6

(d)Find the equation of hyperbola whose eccentricity is 5/2 and focus at (a,0) and its directrix is 4x – 3y – a = 0. — 6

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Answer

Question 1

(a)(i)If $\lambda_1, \lambda_2, \lambda_3$ are all eigenvalues of a matrix A. Then |A| is $\lambda_1.\lambda_2.\lambda_3$

(ii)A linear system AX = B is called consistent if it has a solution.

-- ***(2 Marks)***

(b)We see that A + B and B.A are not exist.

$$A.A` = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 2 \\ 6 & 10 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$$

$$A + 2I = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$|A| = 0 - 2(-2) - 1(-3) = 7.$$

-- ***(8 Marks)***

(c)(i) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 6 = 0$

Then, the eigenvalues are: $\lambda_1 = 3, \quad \lambda_2 = -2.$

From the equation, $\begin{bmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

For : $\lambda_1 = 3$, $\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then – x + y = 0, 4x – 4y = 0

Then x = y = any number except 0

Put y = 1, we get x = 1 and the

corresponding eigenvector is: $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For : $\lambda_2 = -2$, $\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then 4x + y = 0, 4x + y = 0

Then y = – 4x = any number except 0

Put x = 1, we get y = – 4 and the

corresponding eigenvector is: $X_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

-- ***(8 Marks)***

(c)(ii)The eigenvalues of $f(A) = \ln(A^2 + A)$ are $f(3) = \ln 12$, $f(-2) = \ln 2$.

-- ***(2 Marks)***

(c)(iii)Since $T = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$. Then $T^{-1} = -\frac{1}{5}\begin{bmatrix} -4 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$

Then $A^n = T.\begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}.T^{-1} = \begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}.\begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}.\frac{1}{5}\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} 4(3)^n + (-2)^n & (3)^n - (-2)^n \\ 4(3)^n - 4(-2)^n & (3)^n + 4(-2)^n \end{bmatrix}$$

--- ***(5 Marks)***

Question 2

(a) $G = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 2 & 3 \\ -2 & 3 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 2 & -2 & -5 \\ 0 & 3 & 1 & 4 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -4 & -11 \\ 0 & 0 & -2 & -5 \end{array}\right]$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -4 & -11 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We see that rank A is 3 but rank G is 4. Then, there is no solution.

--- ***(4 Marks)***

(b)(i)Since $u_r = (r+1)(2r+1) = 2r^2 + 3r + 1$

Then $S_n = \frac{2}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + n$

Then $S = \infty$

$$S_{10} = \frac{1}{3}(10)(11)(21) + 3(5)(11) + 10 = 945$$

--- ***(3 Marks)***

(b)(ii)Since $u_r = \frac{2}{r^2+4r+3} = \frac{1}{r+1} - \frac{1}{r+3} = [\frac{1}{r+1} - \frac{1}{r+2}] + [\frac{1}{r+2} - \frac{1}{r+3}]$.

Then $S_n = \frac{1}{2} - \frac{1}{n+2} + \frac{1}{3} - \frac{1}{n+3}$.

Then $S = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $S_{10} = \frac{5}{6} - \frac{1}{12} - \frac{1}{13} = \frac{35}{52}$

--- ***(3 Marks)***

(c)(i)Prove that : $1.1! + 2.2! + 3.3! + \cdots + n.n! = (n+1)! - 1$

(1)At n = 1, the left hand side (L.H.S) of this relation is 1 and the right hand side (R.H.S) of this relation is 2! – 1 = 1. Then this relation is true at n = 1.

(2)Assume that this relation is true at n = k.

This means that $1.1! + 2.2! + 3.3! + \cdots + k.k! = (k+1)! - 1$

(3)We shall prove that this relation is true at n = k + 1.

Or prove that, $1.1! + 2.2! + 3.3! + \cdots + (k+1).(k+1)! = (k+2)! - 1$

From step 2, add (k + 1).(k+1)! to both sides, we get

$$1.1! + 2.2! + 3.3! + \cdots k.k! + (k+1).(k+1)! = (k+1)! - 1 + (k+1).(k+1)!$$

Then $1.1! + 2.2! + 3.3! + \cdots k.k! + (k+1).(k+1)! = (k+1)!.(1+k+1) - 1$

$$= (k+2)! - 1$$

Then this relation is true for all n.

-- ***(4 Marks)***

(c)(ii)Prove that: $y^{(n)} = \sin(x + \frac{n\pi}{2})$ where $y = \sin x$

Note that: $\sin(x + \frac{\pi}{2}) = \cos x$

(1)At n = 1, $y^{(1)} = y` = \sin(x + \frac{\pi}{2}) = \cos x$. Then this relation is true at n = 1.

(2)Assume that this relation is true at n = k.

This means that $y^{(k)} = \sin(x + \frac{k\pi}{2})$

(3)We shall prove that this relation is true at n = k + 1.

Or prove that, $y^{(k+1)} = \sin(x + \frac{(k+1)\pi}{2}) = \sin(x + \frac{k\pi}{2} + \frac{\pi}{2}) = \cos(x + \frac{k\pi}{2})$

From step 2, differentiate both sides, we get $y^{(k+1)} = \cos(x + \frac{k\pi}{2})$

Then this relation is true for all n.

-- ***(4 Marks)***

(d)Since $z_1 = 1 + i = re^{i\theta} = \sqrt{2}\, e^{\frac{i\pi}{4}}$.

$$z_2 = -2 + 2i = re^{i\theta} = \sqrt{8}\, e^{\frac{i3\pi}{4}}$$

$$z_1 + z_2 = -1 + 3i\ , \quad z_1 . z_2 = -4$$

$$\sqrt[2]{z_1} = \sqrt[4]{2}\, e^{\frac{i\pi}{8}}$$

$$(z_2)^7 = 8^{\frac{7}{2}}\, e^{\frac{i21\pi}{4}}$$

-- ***(5 Marks)***

(d) $\frac{1}{1-2x+x^2} = \frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots,$ $|x| < 1$

-- ***(2 Marks)***

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Question 3

(a) Determine the locus of the middle points of chords of the parabola $y^2 = 4ax$ which passes through vertex (0, 0).

Answer

Let the middle point of the chord is (h,k) which joining the point $(0,0)$ and (x_1,y_1) on the parabola, the point (x_1,y_1) satisfy the equation of the parabola, hence $y_1^2 = 4ax_1$. The requirement is finding a relation between h and k ,

$$h = \frac{x_1+0}{2} \; and \;\; k = \frac{y_1+0}{2}$$

$$x_1 = 2h \;\; and \;\; y_1 = 2k$$

Substitute in equation of the parabola

$$y_1^2 = 4ax_1$$

$$(2k)^2 = 4a(2h)$$

$$k^2 = 2ah$$

The required equation is $y^2 = 2ax$

(b)Prove that the two lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents two parallel lines if $h^2 = ab$ and $af^2 = bg^2$.

Answer

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Let the two lines are

$lx + my + c_1 = 0$ and $lx + my + c_2 = 0$

Then the equation

$$\left(lx + my + c_1\right)\left(lx + my + c_2\right)$$

$$= l^2x^2 + 2lmxy + m^2y^2 + l(c_1 + c_2)x + m(c_1 + c_2)y + (c_1c_2) = 0$$

Equivalent equation (1)

Compare the coefficients we get

$a = l^2,\ b = m^2,\ h = lm$ and $\boxed{h^2 = ab}$

$$2g = l(c_1 + c_2), \quad 2f = m(c_1 + c_2)$$

$$\frac{g}{f} = \frac{l}{m} \rightarrow \frac{g^2}{f^2} = \frac{l^2}{m^2} = \frac{a}{b} \quad \therefore \boxed{af^2 = bg^2}$$

(c)Find the equation of the bisectors for the lines $2x^2 + xy - 3y^2 - 6x - 14y - 8 = 0$

Answer

Find the separate equation for the two lines

$2x^2+xy-3y^2-6x-14y-8=(x-y+c_1)(2x+3y+c_2)=0$
Compare the coefficient in both sides we find
$2c_1+c_2=-6,$

$3c_1-c_2=-14$
$2x^2+xy-3y^2-6x-14y-8=(x-y-4)(2x+3y+2)=0$
The two lines are $x-y-4=0$ and $2x+3y+2=0$
The equation of the bisector is $\dfrac{x-y-4}{\sqrt{1^2+(-1)^2}}=\pm\dfrac{2x+3y+2}{\sqrt{2^2+3^2}}$
$\dfrac{x-y-4}{\sqrt{2}}=\pm\dfrac{2x+3y+2}{\sqrt{13}} \Rightarrow 13(x-y-4)^2=2(2x+3y+2)^2$
the equation of the bisectors is $\boxed{x^2-10xy-y^2-24x+16y+40=0}$

(d)Show that the equation $x^2+4y^2+4x+8y-8=0$ represent an ellipse. Determine its major, minor axis, foci and its vertices.
Answer
$x^2+4y^2+4x+8y-8=0$
$(x^2+4x+4)+4(y^2+2y+1)=4+4+8$
$(x+2)^2+4(y+1)^2=16$
$\dfrac{(x+2)^2}{16}+\dfrac{(y+1)^2}{4}=1$
$a^2=16,\ b^2=4$
Center at $(-2,-1)$, axis of symmetry $x=-2,\quad y=-1$
Eccentricity $e^2=1-\dfrac{b^2}{a^2}=1-\dfrac{4}{16}=\dfrac{3}{4}$
Foci at $(-2\pm\dfrac{\sqrt{3}}{2},-1)$ and Directrix $x=\pm\dfrac{a}{e}=\pm\dfrac{8}{\sqrt{3}}$

Question (4)
(a)Derive the equation of the chord of the hyperbola $25x^2-16y^2=400$ which bisected at the point $(5,2)$.
Answer
Let the extremes of the chord are (x_1,y_1) and (x_2,y_2)
Then the following relations are satisfied

$x_1 + x_2 = 10$ (1)

$y_1 + y_2 = 4$ (2)

$25x_1^2 - 16y_1^2 = 400$ (3)

$25x_2^2 - 16y_2^2 = 400$ (4)

The equation of the chord is $\dfrac{y-2}{x-5} = \dfrac{y_1 - y_2}{x_1 - x_2}$ (5)

From (3) and (4)

$25(x_1^2 - x_2^2) - 16(y_1^2 - y_2^2) = 0$

$25(x_1 + x_2)(x_1 - x_2) - 16(y_1 + y_2)(y_1 - y_2) = 0$

$25(x_1 + x_2)(x_1 - x_2) - 16(y_1 + y_2)(y_1 - y_2) = 0$

$25(10)(x_1 - x_2) - 16(4)(y_1 - y_2) = 0$

$25(10) - 16(4)\dfrac{(y_1 - y_2)}{(x_1 - x_2)} = 0$

$\therefore \dfrac{(y_1 - y_2)}{(x_1 - x_2)} = -\dfrac{25(10)}{16(4)} = \dfrac{125}{32}$

Substitute in (5) the required equation is $\dfrac{y-2}{x-5} = \dfrac{125}{32}$

(b)Find the equation of the circle which passes through the points (8, 9), (1, 2) and cut the circle $x^2 + y^2 = 25$ at a right

Answer

Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Since the circle cut the circle $x^2 + y^2 = 25$ (2)

Then the line segment of the centers and the two radii form aright angle triangle

$g^2 + f^2 = r_1^2 + r_2^2$

$g^2 + f^2 = 25 + g^2 + f^2 - c \quad \Rightarrow \quad c = 25$

$(8,9)$ on the circle then $8^2 + 9^2 + 16g + 18f + 25 = 0$

$(1,2)$ on the circle then $1^2 + 2^2 + 2g + 4f + 25 = 0$

Simplify the equations we get

$8g + 9f + 85 = 0$

$g + 2f + 15 = 0$

$f = -5$ and $g = -5$

The required equation is $\boxed{x^2 + y^2 - 10x - 10y + 25 = 0}$

(c)Trace the curve $y^2+4x+2y-11=0$.

Answer

The equation written as

$$y^2+4x+2y-11=0$$

$$(y^2+2y+1)+=-4x+12$$

$$(y+1)^2=-4(x-3)$$

Axis of symmetry is $y=-1$ and the perpendicular to it at the vertex is $x=3$ which intersect at the vertex $(3,-1)$, The focal length is $4a=a \Rightarrow a=1$ the curve open to the left side and Focus at $(2,-1)$ directrix equation is $x=4$

(d)Find the equation of hyperbola whose eccentricity is $\mathbf{\frac{5}{2}}$ and focus at $(\boldsymbol{a},\mathbf{0})$ and whose directrix is $\boldsymbol{4x-3y-a}$.

Answer

Let $P(x,y)$ on the curve then the equation is

$$(x-a)^2+y^2=e^2\left(\frac{4x-3y-a}{\sqrt{4^2+3^2}}\right)^2$$

$$x^2-2ax+a^2+y^2=\frac{25}{4}\left(\frac{4x-3y-a}{5}\right)^2$$

$$4\left(x^2-2ax+a^2+y^2\right)=\left(4x-3y-a\right)^2$$

$$4\left(x^2-2ax+a^2+y^2\right)=\left(16x^2+9y^2+a^2-24xy-8xa+6ay\right)$$

$$\boxed{12x^2-16xy+5y^2-8xa+6ay-3a^2=0}$$

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